MODULE 5

Fourier Series and Fourier Transforms

Mathematics-I — MA101

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Syllabus Coverage

Fourier series, Fourier expansion of functions of any period, even and odd functions, half range expansion, Fourier transform and Fourier integral.

Chapter 1: Fourier Series

Fourier series is an infinite series representation of a periodic function in terms of the trigonometric sine and cosine functions. Most single-valued functions occurring in applied mathematics can be expressed in this form, and Fourier series is a powerful method for solving ordinary and partial differential equations, particularly where periodic functions appear as non-homogeneous terms.

Unlike Taylor's series, which requires a function to be continuous and differentiable, a Fourier series can represent functions that are only piecewise continuous — functions with a finite number of jump discontinuities. Because of its periodic nature, a Fourier series constructed over one period remains valid for all values of x.

1.1 Periodic Functions

A function f(x) is said to be periodic with period $T > 0$ if $f(x + T) = f(x)$ for all x, where T is the least such value.

- sin x, cos x are periodic functions with period 2π.
- tan x, cot x are periodic functions with period π.

1.2 Euler's Formulae

The Fourier series for a function f(x) defined in the interval $C \le x \le C + 2\pi$ is given by:

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where the Fourier coefficients (Euler's formulae) are:

$$a_0 = (1/\pi) \int_C^{C+2\pi} f(x)\, dx , \quad a_n = (1/\pi) \int_C^{C+2\pi} f(x) \cos nx\, dx , \quad b_n = (1/\pi) \int_C^{C+2\pi} f(x) \sin nx\, dx$$

1.3 Conditions for Fourier Expansion (Dirichlet's Conditions)

A function f(x) defined on $[0, 2\pi]$ has a valid Fourier series expansion provided:

- f(x) is well defined and single-valued, except possibly at a finite number of points in $[0, 2\pi]$.
- f(x) has a finite number of finite discontinuities in $[0, 2\pi]$.

- f(x) has a finite number of finite maxima and minima.

Note: *These conditions apply equally to functions defined on $[-\pi, \pi]$, $[0, 2l]$, or $[-l, l]$. The trigonometric system $\{1, \cos x, \cos 2x, \ldots, \sin x, \sin 2x, \ldots\}$ forms a complete set of orthogonal functions with common period 2π (similarly $\{1, \cos(\pi x/l), \cos(2\pi x/l), \ldots, \sin(\pi x/l), \ldots\}$ has common period $2l$), and this orthogonality is what allows the Euler formulae to isolate each coefficient.*

1.4 Fourier Series on Different Intervals

Let f(x) be defined on $[0, 2\pi]$ with $f(x + 2\pi) = f(x)$ for all x. Then:

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = (1/\pi)\int_0^{2\pi} f(x)\,dx, \quad a_n = (1/\pi)\int_0^{2\pi} f(x)\cos nx\,dx, \quad b_n = (1/\pi)\int_0^{2\pi} f(x)\sin nx\,dx$$

Let f(x) be defined on $[-\pi, \pi]$ with $f(x + 2\pi) = f(x)$. Then:

$$a_0 = (1/\pi)\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = (1/\pi)\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = (1/\pi)\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Let f(x) be defined on $[0, 2l]$ with $f(x + 2l) = f(x)$. Then:

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} (a_n \cos(n\pi x/l) + b_n \sin(n\pi x/l))$$

$$a_0 = (1/l)\int_0^{2l} f(x)\,dx, \quad a_n = (1/l)\int_0^{2l} f(x)\cos(n\pi x/l)\,dx, \quad b_n = (1/l)\int_0^{2l} f(x)\sin(n\pi x/l)\,dx$$

Let f(x) be defined on $[-l, l]$ with $f(x + 2l) = f(x)$. Then:

$$a_0 = (1/l)\int_{-l}^{l} f(x)\,dx, \quad a_n = (1/l)\int_{-l}^{l} f(x)\cos(n\pi x/l)\,dx, \quad b_n = (1/l)\int_{-l}^{l} f(x)\sin(n\pi x/l)\,dx$$

1.5 Fourier Series for Even and Odd Functions

Let f(x) be defined on $[-\pi, \pi]$ with $f(x + 2\pi) = f(x)$.

Case (i): f(x) is an even function

Since cos nx is even, $f(x)\cdot\cos nx$ is even and $f(x)\cdot\sin nx$ is odd, so $b_n = 0$. The Fourier series contains only cosine terms:

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos nx, \quad a_n = (2/\pi)\int_0^{\pi} f(x)\cos nx\,dx, \; n = 0, 1, 2, \ldots$$

Case (ii): f(x) is an odd function

Since cos nx is even, $f(x)\cdot\cos nx$ is odd, so $a_0 = 0$ and $a_n = 0$. The Fourier series contains only sine terms:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad b_n = (2/\pi)\int_0^{\pi} f(x)\sin nx\,dx$$

1.6 Worked Examples

Example 1 — Find the Fourier series to represent $f(x) = x^2$ in the interval $(0, 2\pi)$.

Solution:

Using Euler's formulae with $f(x) = x^2$:

$$a_0 = (1/\pi)\int_0^{2\pi} x^2\,dx = (1/\pi)[x^3/3]_0^{2\pi} = (8/3)\pi^2$$

Integrating a_n and b_n by parts twice (and using $\sin 2n\pi = 0$, $\cos 2n\pi = 1$):

$$a_n = 4/n^2, \quad b_n = -4\pi/n$$

Hence the Fourier series for $f(x) = x^2$ on $(0, 2\pi)$ is:

$$x^2 = (4\pi^2)/3 + \Sigma_{n=1}^\infty [(4/n^2) \cos nx - (4\pi/n) \sin nx]$$

Example 2 — Find the Fourier series of the periodic function

$$f(x) = -\pi \text{ for } -\pi < x < 0 , \ \ f(x) = x \text{ for } 0 < x < \pi$$

Hence deduce that $1/1^2 + 1/3^2 + 1/5^2 + \cdots = \pi^2/8$.

Solution:

Splitting the integrals over $(-\pi, 0)$ and $(0, \pi)$ and evaluating gives:

$$a_0 = -\pi/2 , \quad a_n = (1/\pi n^2)[(-1)^n - 1] , \quad b_n = (1/n)(1 - 2\cos n\pi)$$

Hence the Fourier series is:

$$f(x) = -\pi/4 - (2/\pi)(\cos x + \cos 3x/3^2 + \cos 5x/5^2 + \cdots) + (3 \sin x - \sin 2x/2 + \sin 3x/3 - \cdots)$$

Deduction: Putting $x = 0$, and using the fact that f is discontinuous there with $f(0) = ½[f(0^-) + f(0^+)] = -\pi/2$, we get:

$$1/1^2 + 1/3^2 + 1/5^2 + \cdots = \pi^2/8$$

Example 3 — Expand $f(x) = x^2$ as a Fourier series in $[-\pi, \pi]$.

Hence deduce that $1/1^2 + 1/2^2 + 1/3^2 + 1/4^2 + \cdots = \pi^2/6$.

Solution:

Since x^2 is even on $[-\pi, \pi]$, $b_n = 0$. Computing a_0 and a_n:

$$a_0 = (2/\pi)\int_0^\pi x^2 \, dx = 2\pi^2/3 , \quad a_n = (4/n^2)(-1)^n$$

Hence:

$$x^2 = \pi^2/3 + 4(-\cos x + \cos 2x/2^2 - \cos 3x/3^2 + \cos 4x/4^2 - \cdots)$$

Deduction: Putting $x = \pi$ ($\cos n\pi = (-1)^n$):

$$\pi^2/6 = 1 + 1/2^2 + 1/3^2 + 1/4^2 + \cdots$$

Example 4 — Expand in a Fourier series:

$$f(x) = 0 \text{ for } -\pi < x < 0 ; \ \ f(x) = \pi - x \text{ for } 0 \le x < \pi$$

Solution:

With $l = \pi$, the coefficients work out (via integration by parts) to:

$$a_0 = \pi/2 , \quad a_n = [1 - (-1)^n]/(n^2\pi) , \quad b_n = 1/n$$

Therefore the Fourier series for $f(x)$ is:

$$f(x) = \pi/4 + \Sigma_{n=1}^\infty \{ [1 - (-1)^n]/(n^2\pi) \cdot \cos nx + (1/n) \sin nx \} , \ (-\pi < x < \pi)$$

Chapter 2: Half Range Fourier Series

Half range series are used when a function is defined only on the half-interval $[0, \pi]$ or $[0, l]$ and needs to be represented purely in sines or purely in cosines — equivalent to extending the function as an odd or even function, respectively, before applying the ordinary Fourier series.

2.1 Half Range Fourier Sine Series on [0, π]

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx , \quad b_n = (2/\pi)\int_0^{\pi} f(x) \sin nx \, dx$$

This is equivalent to the Fourier series of the odd extension of $f(x)$ on $[-\pi, \pi]$.

2.2 Half Range Fourier Cosine Series on [0, π]

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos nx , \quad a_0 = (2/\pi)\int_0^{\pi} f(x) \, dx , \quad a_n = (2/\pi)\int_0^{\pi} f(x) \cos nx \, dx$$

This is equivalent to the Fourier series of the even extension of $f(x)$ on $[-\pi, \pi]$.

2.3 Half Range Fourier Sine Series on [0, l]

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/l) , \quad b_n = (2/l)\int_0^{l} f(x) \sin(n\pi x/l) \, dx$$

2.4 Half Range Fourier Cosine Series on [0, l]

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos(n\pi x/l) , \quad a_0 = (2/l)\int_0^{l} f(x) \, dx , \quad a_n = (2/l)\int_0^{l} f(x) \cos(n\pi x/l) \, dx$$

2.5 Important Formulae

- $\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$ if $f(2a - x) = f(x)$; equals 0 if $f(2a - x) = -f(x)$.
- $\int_{-a}^{a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$ if f is even; equals 0 if f is odd.
- f is even if $f(-x) = f(x)$; f is odd if $f(-x) = -f(x)$.
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$, $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ $(a < c < b)$
- $\int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx$, also $\int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$

Chapter 3: Fourier Integral

Fourier series are powerful for functions that are periodic, or of interest only on a finite interval. Many problems, however, involve nonperiodic functions of interest on the whole x-axis. Extending the method of Fourier series to such functions leads to the Fourier integral.

3.1 Fourier Integral Theorem

The Fourier integral theorem states:

$$f(x) = (1/\pi) \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos u(t - x) \, dt \, du$$

Proof (sketch): Starting from the Fourier series of $f(x)$ on $(-c, c)$ and substituting the Euler formulae, one obtains:

$$f(x) = (1/2c)\int_{-c}^{c} f(t) \left\{ 1 + 2\sum_{n=1}^{\infty} \cos[(n\pi/c)(t - x)] \right\} dt$$

Letting $c \to \infty$ and setting $n\pi/c = \omega$ (so $\pi/c = d\omega$), the sum becomes an integral over ω, giving:

$$f(x) = (1/\pi) \int_0^\infty \int_{-\infty}^\infty f(t) \cos \omega(t - x)\, dt\, d\omega \qquad \text{(Proved)}$$

3.2 Fourier Sine and Cosine Integrals

Expanding $\cos \omega(t - x) = \cos \omega t \cos \omega x + \sin \omega t \sin \omega x$ in the Fourier integral theorem and separating cases by symmetry of f(t) gives:

$$f(x) = (2/\pi) \int_0^\infty \cos \omega x\, d\omega \int_0^\infty f(t) \cos \omega t\, dt \qquad \text{(Fourier Cosine Integral — valid when } f \text{ is even)}$$

$$f(x) = (2/\pi) \int_0^\infty \sin \omega x\, d\omega \int_0^\infty f(t) \sin \omega t\, dt \qquad \text{(Fourier Sine Integral — valid when } f \text{ is odd)}$$

3.3 Fourier's Complex Integral

Adding i times the (vanishing) sine-integral term to the Fourier integral theorem and using $\cos \theta + i \sin \theta = e^{i\theta}$ gives the complex form:

$$f(x) = (1/2\pi) \int_{-\infty}^\infty e^{-i\omega x}\, d\omega \int_{-\infty}^\infty f(t) e^{i\omega t}\, dt$$

3.4 Worked Example

Example 5 — Express f(x) = 1 for |x| ≤ 1, f(x) = 0 for |x| > 1, as a Fourier integral.

Hence evaluate $\int_0^\infty (\sin u \cos ux)/u\, du$.

Solution:

Using the Fourier integral theorem with f(t) = 1 on [−1, 1] and 0 elsewhere, and integrating with respect to t:

$$f(x) = (1/\pi) \int_0^\infty [\sin u(1 - x) + \sin u(1 + x)]/u\, du$$

Using $\sin C + \sin D = 2 \sin[(C+D)/2] \cos[(C-D)/2]$ and simplifying:

$$f(x) = (2/\pi) \int_0^\infty (\sin u \cos ux)/u\, du \quad \Longrightarrow \quad \int_0^\infty (\sin u \cos ux)/u\, du = (\pi/2) f(x)$$

Therefore:

- For $x < 1$: $\int_0^\infty (\sin u \cos ux)/u\, du = \pi/2$
- For $x > 1$: $\int_0^\infty (\sin u \cos ux)/u\, du = 0$
- For x = 1 (point of discontinuity of f): value = $(\pi/2 + 0)/2 = \pi/4$

Chapter 4: Fourier Transforms

The Fourier transform decomposes a waveform — a function of time (or space) — into the frequencies that compose it. It is a generalization of the Fourier series that extends the idea to nonperiodic functions, allowing any suitable function to be viewed as a superposition of simple sinusoids. Fourier transforms are especially useful for solving partial differential equations such as the heat and wave equations.

4.1 Definition of the Fourier Transform

Starting from Fourier's complex integral and writing ω = s, and defining $F(s) = (1/\sqrt{2\pi})\int_{-\infty}^\infty f(t) e^{ist}\, dt$, we obtain the Fourier transform pair:

$$F(s) = F[f(x)] = (1/\sqrt{2\pi}) \int_{-\infty}^\infty f(t) e^{ist}\, dt \qquad \text{(Fourier Transform)}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} F(s)\, ds \qquad \text{(Inverse Fourier Transform)}$$

4.2 Fourier Sine Transform

From the Fourier sine integral, setting $F(s) = \sqrt{2/\pi}\int_0^\infty f(t) \sin st\, dt$ gives the transform pair:

$$F(s) = F_s[f(x)] = \sqrt{2/\pi} \int_0^\infty f(t) \sin st\, dt$$

$$f(x) = \sqrt{2/\pi} \int_0^\infty F(s) \sin sx\, ds \qquad \text{(Inverse Fourier Sine Transform)}$$

4.3 Fourier Cosine Transform

Similarly, from the Fourier cosine integral:

$$F(s) = F_c[f(x)] = \sqrt{2/\pi} \int_0^\infty f(t) \cos st\, dt$$

$$f(x) = \sqrt{2/\pi} \int_0^\infty F(s) \cos sx\, ds \qquad \text{(Inverse Fourier Cosine Transform)}$$

4.4 Some Important Standard Integrals

- $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2} \cdot (a \sin bx - b \cos bx)$
- $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2} \cdot (a \cos bx + b \sin bx)$
- $\int_0^\infty \frac{\sin ax}{x}\, dx = \pi/2$
- $\int_0^\infty e^{-x^2}\, dx = \sqrt{\pi}/2$
- $\int_{-\infty}^{\infty} \frac{\sin mx}{(x-b)^2+a^2}\, dx = (\pi/a)\, e^{-am} \sin bm$, $[m > 0]$

4.5 Worked Examples

Example 6 — Find the Fourier transform of e^{-ax^2}, where $a > 0$.

Solution:

Completing the square in the exponent $-ax^2 + isx$ and substituting $u = x\sqrt{a} - is/(2\sqrt{a})$:

$$F\{e^{-ax^2}\} = e^{-s^2/4a} / \sqrt{2\pi a} \cdot \int_{-\infty}^{\infty} e^{-u^2}\, du = e^{-s^2/4a} / \sqrt{2a}$$

Example 7 — Find the Fourier transform of $f(x) = 2$ for $|x| < a$, $f(x) = 0$ for $|x| > a$.

Solution:

Direct integration gives:

$$F\{f(x)\} = (2/\sqrt{2\pi})\int_{-a}^{a} e^{isx}\, dx = (4/\sqrt{2\pi}\cdot s) \sin as = 2\sqrt{2/\pi} \cdot (\sin sa)/s$$

Example 8 — Find the Fourier sine transform of $f(x) = 1/x$.

Solution:

Substituting $sx = t$:

$$F_s[1/x] = \sqrt{2/\pi} \int_0^\infty (\sin t)/t\, dt = \sqrt{2/\pi} \cdot (\pi/2) = \sqrt{\pi/2}$$

Example 9 — Find the Fourier sine transform of e^{-ax}.

Solution:

Using $\int_0^\infty e^{ax} \sin bx\, dx = e^{ax}/(a^2+b^2)(a \sin bx - b \cos bx)$:

$$F_s[e^{-ax}] = \sqrt{2/\pi} \cdot s/(a^2 + s^2)$$

Example 10 — Find the Fourier cosine transform of $f(x) = 5e^{-2x} + 2e^{-5x}$.

Solution:

Applying the cosine transform to each exponential term separately:

$$F_c[f(x)] = 10 \left[1/(s^2+4) + 1/(s^2+25) \right]$$

4.6 Properties of Fourier Transforms

Linearity Property

If $F_1(s)$ and $F_2(s)$ are the Fourier transforms of $f_1(x)$ and $f_2(x)$ respectively, and a, b are constants:

$$F[a\, f_1(x) + b\, f_2(x)] = a\, F_1(s) + b\, F_2(s)$$

Change of Scale Property

$$F[f(ax)] = (1/a)\, F(s/a)$$

Shifting Property

$$F[f(x - a)] = e^{isa}\, F(s)$$

4.7 Fourier Transform of Derivatives

For the n-th derivative of f(x):

$$F[f^{(n)}(x)] = (-is)^n\, F(s)$$

In particular, for the second derivative:

$$F[f''(x)] = (-is)^2\, F[f(x)] = -s^2\, \bar{f}\text{, where } \bar{f} \text{ is the Fourier transform of } f$$

If F_c and F_s denote the cosine and sine Fourier transforms of f(x), then (assuming $f(x) \to 0$ as $x \to \infty$):

$$F_c[f'(x)] = -\sqrt{2/\pi}\, f(0) + s\, F_s(s)$$

4.8 Fourier Transform of Partial Derivatives

For a function u(x, t), the Fourier transform (in x) of its second partial derivative is:

$$F[\partial^2 u/\partial x^2] = -s^2\, F(u)$$

The corresponding Fourier sine and cosine transforms of the second partial derivative are:

$$F_s[\partial^2 u/\partial x^2] = s \cdot (u)^{x}{}_{=0} - s^2\, F_s(u)$$

$$F_c[\partial^2 u/\partial x^2] = -(\partial u/\partial x)^{x}{}_{=0} - s^2\, F_c(u)$$

These identities are the key tool for converting a linear PDE in x and t into an ordinary differential equation in t, which can then be solved and inverted back.

Chapter 5: Practice Problems

Attempt the following problems using the definitions and properties developed in this module.

- 1) Find the Fourier Transform of $f(x)$ if $f(x) = x$ for $|x| \le a$, and $f(x) = 0$ for $|x| > a$.
- 2) Show that the Fourier Transform of $f(x) = a - |x|$ for $|x| < a$, and 0 for $|x| > a > 0$, is $\sqrt{(2/\pi)} \cdot (1 - \cos as)/s^2$. Hence show that $\int_0^\infty (\sin t / t)^2 \, dt = \pi/2$.
- 3) Show that the Fourier Transform of $f(x) = \sqrt{(2\pi)}/(2a)$ for $|x| \le a$, and 0 for $|x| > a$, is $(\sin sa)/(sa)$.
- 4) Find the Fourier cosine Transform of e^{-ax}.
- 5) Find the Fourier transform of $F(x) = x^2$ for $|x| < a$, and 0 for $|x| > a$.
- 6) Find the Fourier Sine Transform of $f(x) = 1/[x(x^2 + a^2)]$.
- 7) Find the Fourier Sine and Cosine Transform of $a e^{-\alpha x} + b e^{-\beta x}$, $\alpha, \beta > 0$.
- 8) Find $f(x)$ if its Fourier Sine transform is $s/(1 + s^2)$.
- 9) Find $f(x)$ if its Fourier Sine transform is $(2\pi s)^{1/2}$.

Answers

Q. No.	Answer
1	$(1/\sqrt{2\pi}) \cdot 2i/s^2$
4	$F_c\{f(x)\} = \sqrt{(2/\pi)} \cdot a/(a^2 + s^2)$
5	$(2a^2/s - 4/s^2) \sin as + (4a/s^2) \cos as$
6	$(\pi/2a^2)(1 - e^{-ax})$
7	Sine: $as/(s^2+\alpha^2) + bs/(s^2+\beta^2)$ | Cosine: $a\alpha/(s^2+\alpha^2) + b\beta/(s^2+\beta^2)$
8	$2 \sin^2(ax) / (\pi^2 x^2)$
9	$1/(x\sqrt{x})$

Quick Reference — All Key Formulae (Module 5)

Fourier Series on [−π, π]

$f(x) = a_0/2 + \Sigma(a_n \cos nx + b_n \sin nx)$; $a_0, a_n, b_n = (1/\pi)\int_{-\pi}^{\pi} f(x)\{1, \cos nx, \sin nx\}\, dx$

Fourier Series on [−l, l]

$f(x) = a_0/2 + \Sigma(a_n \cos(n\pi x/l) + b_n \sin(n\pi x/l))$; $a_0, a_n, b_n = (1/l)\int_{-l}^{l} f(x)\{1, \cos(n\pi x/l), \sin(n\pi x/l)\}\, dx$

Even / Odd Function Series

Even: $f(x) = a_0/2 + \Sigma\, a_n \cos nx$, $a_n = (2/\pi)\int_0^{\pi} f \cos nx\, dx$. Odd: $f(x) = \Sigma\, b_n \sin nx$, $b_n = (2/\pi)\int_0^{\pi} f \sin nx\, dx$

Half Range Series on [0, l]

Sine: $f(x) = \Sigma\, b_n \sin(n\pi x/l)$, $b_n = (2/l)\int_0^{l} f \sin(n\pi x/l)\, dx$. Cosine: $f(x) = a_0/2 + \Sigma\, a_n \cos(n\pi x/l)$, $a_n = (2/l)\int_0^{l} f \cos(n\pi x/l)\, dx$

Fourier Integral Theorem

$$f(x) = (1/\pi) \int_0^{\infty}\int_{-\infty}^{\infty} f(t) \cos \omega(t - x)\, dt\, d\omega$$

Fourier Sine / Cosine Integrals

$f(x) = (2/\pi)\int_0^{\infty} \sin \omega x\, d\omega \int_0^{\infty} f(t) \sin \omega t\, dt$; $f(x) = (2/\pi)\int_0^{\infty} \cos \omega x\, d\omega \int_0^{\infty} f(t) \cos \omega t\, dt$

Fourier Transform Pair

$$F(s) = (1/\sqrt{2\pi})\int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt \iff f(x) = (1/\sqrt{2\pi})\int_{-\infty}^{\infty} e^{-isx} F(s)\, ds$$

Fourier Sine / Cosine Transform Pairs

$$F_s(s) = \sqrt{2/\pi}\int_0^{\infty} f(t) \sin st\, dt \iff f(x) = \sqrt{2/\pi}\int_0^{\infty} F(s) \sin sx\, ds$$

$$F_c(s) = \sqrt{2/\pi}\int_0^{\infty} f(t) \cos st\, dt \iff f(x) = \sqrt{2/\pi}\int_0^{\infty} F(s) \cos sx\, ds$$

Properties of Fourier Transform

Linearity: $F[af_1+bf_2] = aF_1+bF_2$ Scaling: $F[f(ax)] = (1/a)F(s/a)$ Shifting: $F[f(x-a)] = e^{isa}F(s)$

Fourier Transform of Derivatives

$$F[f^{(n)}(x)] = (-is)^n F(s)\ ;\quad F_c[f'(x)] = -\sqrt{2/\pi}\, f(0) + sF_s(s)$$

— End of Module 5 Notes —